The Structure of Symmetry Groups

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Abstract—*This paper goes for taking care of the issues and talking about the properties of dihedral groups Dn. It has dependably been a troublesome undertaking in deciding the practices of reflections and rotational symmetries in these symmetry groups .We thusly concentrate the nature and properties of these symmetry components including the conjugacy class measure in Dn. It is found that the conjugacy classes of Dn, where the connection "Conjugacy" is a proportionality connection, decides the whole structure of the symmetry groups. The representations of the conjugacy class size of Dn uncovers that the order of the centers of Dn are 1 (for n-odd) and 2 (for n-even), and thusly, prompting to investigation of two diverse class equations of Dn*

Keywords*: Conjugacy Class, Center, Class Equation ,Rotations, Reflections*

1. INTRODUCTION

In group hypothesis, a dihedral group is the group of symmetries of a consistent polygon, including both turns and reflections (Dummit, 2004). Dihedral groups are among the easiest cases of limited groups, and they assume an essential part in group hypothesis, geometry and science. There are two contending documentations for the dihedral bunches related to a polygon with n sides. Here Dn alludes to the symmetries of a consistent polygon with n sides.

2. ELEMENTS OF THE GROUPS DN

A general polygon with n sides has 2n distinct symmetries: n rotational symmetries and n reflection symmetries. The related pivots and reflections make up the dihedral amass Dn. On the off chance that n is odd, every pivot of symmetry associates the midpoint of one side to the inverse vertex. On the off chance that n is even, there are n/2 axes of symmetry associating the midpoints of inverse sides and n/2 axes of symmetry interfacing inverse vertices. In either case, there are n axes of symmetry inside and out and 2n components in the symmetry group.

3. GROUP STRUCTURE OF D_N

The structure of two symmetries of a standard polygon is again symmetry, as on account of geometric protest. The arrangement operation is not commutative, and by and large, the group Dn has the accompanying components:

$D_n=\{r_0 = e, r_1, r_2,\dots, r_{n-1}, f_0,f_2, \dots, f_{n-1}\}$ with the following properties:

$r_i\, r_j = r_{(i+j)modn}$;

$r_i\, f_j = f_{(i-j)modn}$;

$f_i\, f_j = r_{(i-j)modn}$.

The 2n elements of D_n can be written as e, $r,r^2, \dots, r^{n-1}$, f, rf, $r^2 f, \dots, r^{n-1}f$. The first n elements are the elements of the rotations and the remaining n elements are axes reflections (all have order 2). Obviously, the product of two rotations or two reflections is a rotation, while the product of a rotation and a reflection is a reflection. From the information provided so far on D_n, it is therefore convenient to write D_n as

$$D_n = \langle r, f \mid r^n = e = f^2 , f rf = r^{-1} , rfr = f \rangle \qquad 1$$

The group with representation as in equation 1 above or as

$$D_n = \langle x,y|x^2 = y^2 = (xy) = e\rangle \qquad 2$$

From the second presentation, it follows that D_n belongs to the class of Coxeter groups.

4. CONJUGACY CLASSES IN DN

Let G be any group. Two elements α and σ of G are said to be conjugate if $\alpha = \gamma\sigma\gamma^{-1}$ for some γ∈G (Samaila, 2010). In other words, if σ, γ ∈G, we define the conjugate of σ by γ or σ by γ^{-1} to be the element $\gamma\sigma\gamma^{-1}$ or $\gamma^{-1}\sigma\gamma$ respectively.

Proposition 1: Let G be a group, and define the relation ∼ on G by α∼σ if α and σ are conjugate in G. Then ∼ is an equivalence relation (Bianchi, 2001). Since the relation ∼ is an equivalence relation on G, its equivalence classes partition G. The equivalence classes under this relation are called the conjugacy classes of G. Hence the conjugacy class of α∈G is given by

$$[\alpha] = \{\gamma\alpha\gamma^{-1}|\gamma\in G\}.$$

Simple Results

1. Let G be any group and let x, $g_1, g_2, \dots, g_n$∈G. Then for any n, the conjugate of $g_1g_2\dots g_n$ by x is the product of the conjugate by x of $g_1, g_2, \dots, g_n$.

Proof: The conjugate of $g_1g_2\dots g_n$ by x is given by

$$x(g_1g_2\dots g_n)x^{-1}= (xg_1x^{-1})(xg_2x^{-1})\dots(xg_nx^{-1}) \qquad 3$$

Where xg_ix^{-1} is the conjugate of each g_i , $1 \leq i \leq n$, by x. Hence the result follows.∗

2. Let G be an abelian group. Then for any $\alpha \in G$, the conjugacy class of α is the singleton set {α}.

Proof: Let G be an abelian group. Let $\alpha, \gamma \in G$. Then the conjugate of α by γ is $\gamma\alpha\gamma^{-1}$. Now, the conjugacy class of α is given by

$[\alpha] = \{\gamma\alpha\gamma^{-1} | \gamma \in G\}$ by definition. Now,

$[\alpha] = \{(\gamma\alpha)\gamma^{-1} | \gamma \in G\}$,

$[\alpha] = \{(\alpha\gamma)\gamma^{-1} | \gamma \in G\}$

(G is abelian),

$[\alpha] = \{\alpha(\gamma\gamma^{-1}) | \gamma \in G\}$

$[\alpha] = \{\alpha(e) | \gamma \in G\} = \{\alpha\}$.∗

Note that this is true for all abelian groups but not non-abelian groups**.**

Lemma 2: Let G be a group and let $\alpha, \beta \in G$. If α and β are conjugate, then α and β have the same order (Arid, 2004).

Proof: Since α and β are conjugate to each other, there exists an element $\gamma \in G$ such that $\alpha = \gamma\beta\gamma^{-1}$. Let $o(\alpha) = m$ and $o(\beta) = n$ for some positive integers m and n. Now, by definition, $e = \alpha^m = (\gamma\beta\gamma^{-1})(\gamma\beta\gamma^{-1})\ldots(\gamma\beta\gamma^{-1})$ m-times

$= \gamma\beta^m\gamma^{-1}$

This means that $\gamma^{-1} e\gamma = \beta^m$, i.e. $\beta^m = e$. Thus, n|m.

Similarly, $e = \beta^n = (\gamma^{-1}\alpha\gamma)(\gamma^{-1}\alpha\gamma)\ldots(\gamma^{-1}\alpha\gamma)$ n-times

$= \gamma^{-1}\alpha^n\gamma$

i.e. $\gamma e\gamma^{-1} = \alpha^n$ or $\alpha^n = e$. Thus, m|n, and hence, m = n.

Remark: Consider the symmetric group S_3 = {(1), (1 2), (1 3), (2 3), (1 2 3), (1 3 2)}. Now we have the conjugacy classes in S_3 as follows:

C_{S3} = {(1)} ;

C_{S3} ={(23), (13), (12)} ;

C_{S3} = {(123) ,(132)} .

In the conjugacy classes above, the second conjugacy class consists precisely of the elements of order 2 in the symmetric group S_3, and the third conjugacy class consists precisely of the elements of order 3. But this doesn't always happen quite so nicely, but it is true that conjugate elements do have the same order.

5. DETERMINATION OF CONJUGACY CLASSES IN D_N

Every one of the appearance in Dn are conjugate to each other if n is odd, yet they fall into two conjugacy classes for n even. In the event that we think about the isometries of a consistent n-gon: for odd n there are turns in the group between each match of mirrors, while for even n just 50% of the mirrors can be come to from one by these pivots. Geometrically, in an odd polygon each hub of symmetry goes through a vertex and a side, while in an even polygon a large portion of the axes go through two vertices, and half go through two sides.

Logarithmically, this is an occurrence of the conjugate Sylow hypothesis (for n odd): for n odd, every reflection, together with the personality, frame a subgroup of request 2, which is a Sylow 2-subgroup (2 = 21 is the most extreme force of 2 separating 2n = 2(2k + 1)), while for n even, these request 2 subgroups are not Sylow subgroups since 4 (a higher force of 2) partitions the request of the group. Rather, for n even there is an external automorphism trading the two sorts of reflections (appropriately, a class of external automorphisms which are all conjugate by an inward automorphism). On account of a turn, every pivot is conjugate just to its opposite (which is an alternate revolution aside from 1 and, for even n, $\alpha^{n/2}$

Theorem 5: The conjugacy classes in D_n are as follows, depending on the parity of n (Beltran, 2006).

1. Odd n: {1}, $\{\alpha^{\pm 1}\}$, $\{\alpha^{\pm 2}\}$, …, $\{\alpha^{\pm(n-1)/2}\}$, $\{\alpha^i\beta : 0 \leq i \leq n-1\}$.

2. Even n: {1}, $\{\alpha^{\pm 1}\}$, $\{\alpha^{\pm 2}\}$, …, $\{\alpha^{\pm(n-2)/2}\}$, {α n/2 }, $\{\alpha^{2i}\beta : 0 \leq i \leq (n-2)/2\}$ and $\{\alpha^{2i+1}\beta : 0 \leq i \leq (n-2)/2\}$.

Proof: Every rotation in D_n is conjugate to its inverse, since

$\beta\alpha^j\beta = \alpha^{-j}$.

More generally, the formulas

$\alpha^i \alpha j \alpha^{-i} = \alpha^j$; $(\alpha^i\beta)\alpha^j(\alpha^i\beta)^{-1} = \alpha^{-j}$; $0 \leq i \leq n-1$, $0 \leq j \leq n-1$,

As i varies, it shows that the only conjugates of α^j in D_n are α^j and α^{-j} .

To find the conjugacy class of β, we compute

$\alpha^i\beta\alpha^{-i} = \alpha^{2i \bmod n}\beta$;

$(\alpha^i\beta)\beta(\alpha^i\beta)^{-1} = \alpha^{2i \bmod n}\beta$;

$1 \leq i \leq n$.

As i varies, $\alpha^{2i}\beta$ runs through the reflections in which α occur with an exponent divisible by 2. If n is odd then every integer modulo n is a multiple of 2 (since 2 is invertible mod n so we can solve $a \equiv 2x \bmod n$ given a). Therefore

$\{\alpha^{2i}\beta : i \in Z\} = \{\alpha^i\beta : i \in Z\}$,

So every reflection in D_n is conjugate to β for odd n. When n is even, however, we only get half the reflections, i.e.

$\{\beta, \alpha^2\beta, \alpha^4\beta, \ldots, \alpha^{n-2}\beta\}$ as conjugates of β, demonstrated by Samaila, 2010 using D_{12}. The other half are conjugate to αβ in the following manner:

$\alpha^i(\alpha\beta)\alpha^{-i} = \alpha^{(2i+1) \bmod n}\beta$;

$(\alpha^i \beta)(\alpha\beta)(\alpha^i \beta)^{-1} = \alpha^{(2i-1) \bmod n} \beta$.

As i varies, this gives us

$\{\alpha\beta, \alpha^3 \beta, \ldots, \alpha^{n-1}\beta\}$.

6. REPRESENTATION OF THE CONJUGACY CLASSES IN D_N

Considering the definition of Conjugacy class explained above, if we represent the elements of D_n as $\{I, \alpha, \alpha^2, \ldots, \alpha^{n-1}, \alpha^i \beta;\ 0 \le i \le n-1\}$ where each element is to represent a conjugacy class, then we shall have the size of the conjugacy classes. Again, there are (n-1)/2 pairs of conjugate rotations when n is odd (exclude the identity) and (n-2)/2 pairs of conjugate rotations for even n (exclude the identity and $\alpha^{n/2}$). In both cases, whether n is even or odd, the sum of the sizes of the conjugacy classes in D_n equals 2n.

Reprt.	I	α	α^2……………	$\alpha^{(n-1)/2}$	β
Size	1	2	2 ………………	2	n

Table 1: Conjugacy class representation in D_n for n odd

Reprt.	I	α	α^2……………	$\alpha^{(n-2)/2}$	$\alpha^{n/2}$	β	$\alpha\beta$
Size	1	2	2 …………….	2	1	n/2	n/2

Table 2: Conjugacy class representation in D_n for n even

⅔ Obviously, in table 1 and 2 of the representations of the conjugacy classes of D_n above, the sum of the sizes of the conjugacy classes amounted to 2n.

7. CENTER OF D_N

Recall that the centralizer of the subgroup H in a group D_n is the set of elements of D_n which commute with every elements of H, namely

$C(H)_{D_n} = \{g \in D_n \mid \alpha g = g\alpha \text{ for all } \alpha \in H\}$. Hence, the centralizer of the subgroup H of the group D_n is the subgroup H itself if H represent the set of all rotations (including identity) in D_n. Again the center of the group D_n is the subgroup of D_n defined by $Z(D_n) = \{g \in D_n : gh = hg\ \forall\ h \in D_n\}$. Thus, $Z(D_n) = \{I\}$, the trivial subgroup if n is odd and $Z(D_n) = \{I, \alpha^{n/2}\}$ if n is even. The center of any group G is a normal subgroup of G. Hence, we have the following results.

Theorem 6: The center of D_n $(n \ge 3)$ is trivial when n is odd and $\{I, \alpha^{n/2}\}$ when n is even (Beltran, 2006).

Proof: This follows immediately from tables 1 and 2 above, since the center is the set of elements which are in conjugacy classes of size 1.∗

Corollary 7: If $n \ge 6$ is twice an odd number then $D_n \cong D_{n/2} \times Z/(2)$.

Proof: Let $H = \langle \alpha^2, \beta \rangle \cong D_{n/2}$ and $Z = \{I, \alpha^{n/2}\}$. Then Z is normal in D_n, HZ is a subgroup of D_n and the elements of H commute with the elements of Z. Let $f : H \times Z \to D_n$ be a function defined by $f(h, z) = hz$ for all $h \in H$, $z \in Z$. This function is a homomorphism since Z is the center of D_n. The kernel is $H \cap Z$, which is trivial. That is, $\alpha^{n/2} \notin H$. Indeed, if $\alpha^{n/2} \in H$ then either $\alpha^{n/2} = \alpha^{2i}$ or $\alpha^{n/2} = \alpha^{2i}\beta$ for some i. The first condition implies $n/2 \equiv 2i \bmod n$, which is impossible since 2i and the modulus n are even but n/2 is odd. The second condition is also impossible since it implies that β is a power of α. Since f is injective and $O(H \times Z) = 2n = O(D_n)$, f is an isomorphism.

Remark: When n is divisible by 4, then there is no isomorphism in the Corollary above. This is because n and n/2 will be even and the center of D_n will be a cyclic group of size 2 and the center of $D_{n/2} \times Z/(2)$ is a direct product of two cyclic groups of size 2. Hence, the centers are not isomorphic, so D_n is not isomorphic to $D_{n/2} \times Z/(2)$.

8. THE CLASS EQUATION OF A FINITE GROUP

Given any finite group G, let Z(G) be the center of G. Then $G \cap \{Z(G)\}^c$ is a disjoint union of conjugacy classes. Let m be the number of conjugacy classes contained in $G \cap \{Z(G)\}^c$, and let $i_1, i_2, \ldots, i_m$ be the number of elements in these conjugacy classes. Then $i_j > 1$ for all j, since the centre Z(G) of G is the subgroup of G consisting of those elements of G whose conjugacy class contains just one element, see tables 1 and 2 above. Now the group G is the disjoint union of its Conjugacy classes, and therefore,

$$|G| = |Z(G)| + i_1 + i_2 + \ldots + i_m \quad (5)$$

This equation is referred to as the class equation of the group G.

Remark: From tables 1 and 2 above, we have $Z(D_n) = \{I\}$ when n is odd and

$Z(D_n) = \{I, \alpha^{n/2}\}$ when n is even, therefore the class equations of D_n can be written as

$|D_n| = |Z(D_n)| + siz(\alpha) + siz(\alpha^2) + \ldots + siz(\alpha^{(n-1)/2}) + siz(\beta)$ (for n-odd)

$= 1 + 2 + 2 + \ldots + 2 + n$; $(2 + 2 + \ldots$ (n-1)/2 times)

And

$|D_n| = |Z(D_n)| + siz(\alpha) + siz(\alpha^2) + \ldots + siz(\alpha^{(n-2)/2}) + siz(\beta) + siz(\alpha\beta)$ (for n-even)

$= 2 + 2 + 2 + \ldots + 2 + n/2 + n/2$; $(2 + 2 + \ldots$ (n/2) times)

9. SUMMARY

The group structures of Dn were inspected and its Conjugacy classes. It was found that in Dn, its components are apportioned into two disjoint sets (one comprises of turns and the other for the reflections) of a similar request (Samaila, 2010). Every pivot is conjugate to its converse, taking note of that for the personality component I and the turn $\alpha^{n/2}$ (for n-even), each is conjugate to itself. For n-odd, the reflection β is conjugate to each different reflections while for n-even, β is conjugate to half of the reflections while the reflection αβ is conjugate to the staying half of the reflections. We have additionally observed that the connection "Conjugacy" is an identicalness connection. The focal point of Dn is observed to be the inconsequential subgroup {I} when n is odd and {I, $\alpha^{n/2}$ } when n is even. Lastly, two class conditions for Dn were determined.

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